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THE INTERFACE CRACK IN ANISOTROPIC BODIES. STRESS SINGULARITIES AND INVARIANT INTEGRALS†

S. A. NAZAROV

St Petersburg

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For the problem of the deformation of a composite anisotropic plate with a crack (in a linear formulation, with no assumption of symmetry), all possible power solutions are listed and general relations between the ordinary and singular solutions are revealed. The asymptotic form of the increment of the potential energy of deformation is computed for the cases of the rectilinear propagation of the crack, deviation of a shoot or branching. The form obtained for the final formula is the same as the classical version of the Griffiths formula and involves two invariant integrals. Two methods of determining the modes of radical singularities of the stress–strain state near the crack tip, associated with the use of force and energy criteria, are proposed.

1. STATEMENT OF THE PROBLEM

Let the lower and upper half-planes $\mathbf{R}^2_\pm$ be filled with homogeneous anisotropic, generally speaking different, materials and meet along the ray Λ_+; here $\Lambda_\pm\ \{x=(x_1,x_2)\in\mathbf{R}^2: x_2=0, \pm x_1>0\}$, $\mathbf{R}^2_\pm=\{x: \pm x_2>0\}$. As we know, for many purposes it is sufficient to study the power solutions (PS)

$$U(x)=r^\Lambda\Phi(\varphi,\ln r) \tag{1.1}$$

of the corresponding homogeneous equations of equilibrium in $\mathbf{R}^2_\pm$, the boundary conditions on Λ_- and the interface (contact) conditions on Λ_+. In (1.1) (r,φ) are polar coordinates on $\mathbf{R}^2\backslash\Lambda_-$ and $r=|x|$, $\varphi\in(-\pi,\pi)$; $\lambda\in\mathbf{C}$, $l\mapsto\Phi(\varphi,l)$ are vector polynomials. We emphasize that u is a three-dimensional vector; the spatial problem is not assumed to split into independent plane and antiplane problems.

We first introduce the necessary notation. By $u^\pm=(u_1^\pm,u_2^\pm,u_3^\pm)$ we mean the displacement vector in $\mathbf{R}^2_\pm$; $u_3^\pm$ is the warping. We denote the stress tensors by $\sigma^\pm(u^\pm)=(\sigma^\pm_{jk}(u^\pm))$, where $j,k=1,2,3$. Furthermore, $\partial_p=\partial/\partial x_p$ and $L^\pm(\nabla)u^\pm$, $N^\pm(\nabla)u^\pm$ are vectors, with projections onto the x_j axis of the form

$$-\partial_1\sigma^\pm_{1j}(u^\pm)-\partial_2\sigma^\pm_{2j}(u^\pm),\ \ \sigma^\pm_{2j}(u^\pm) \tag{1.2}$$

In a Cartesian representation $L^\pm(\nabla)$ and $N^\pm(\nabla)$ are 3×3 matrix differential operators of second and first orders. Finally, $\varepsilon(u)$ is a stress tensor with components $\varepsilon_{jk}(u)$ and, if $\Omega_\pm\subset\mathbf{R}^2_\pm$, then $E^\pm(u^\pm,u^\pm;\Omega_\pm)$ is the elastic energy stored by the body $\Omega_\pm$ under deformation $\varepsilon(u^\pm)$, that is

$$E^\pm(u^\pm,\ v^\pm;\Omega_\pm)=\frac{1}{2}\int_\Omega\sum_{j,k=1}^{3}\varepsilon_{jk}(u^\pm)\sigma_{jk}(u^\pm)dx=E^\pm(v^\pm,u^\pm;\Omega_\pm) \tag{1.3}$$

$$E(u,v;\Omega)=E^+(u^+,\ v^+;\Omega\cap\mathbf{R}^2_+)+E^-(u^-,\ v^-;\Omega\cap\mathbf{R}^2_-) \tag{1.4}$$

Thus, the PS (1.1) satisfies the relations

$$L^\pm(\nabla)U^\pm(x)=0,\ \ x\in\mathbf{R}^2_\pm \tag{1.5}$$

$$\mp N^\pm(\nabla)U^\pm(x)=0,\ \ x\in\Lambda_- \tag{1.6}$$

$$U^+(x)=U^-(x),\ \ N^+(\nabla)U^+(x)=N^-(\nabla)U^-(x),\ \ x\in\Lambda_+ \tag{1.7}$$

The contraction of U onto $\mathbf{R}^2_\pm$ is denoted by $U^\pm$, and the traced $U^\pm$ at $x_2=0$ is taken as the limit as $x_2\to\pm0$. For any sufficiently smooth vector functions u and υ we have the Betti identity

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$$\int_{\Omega} v\, Lu\, dx + \int_{\partial\Omega} v\, \sigma^{(\nu)}(u)\, ds + \int_{\Lambda_+ \cap \partial\Omega} v\,(N^+u^+ - N^-u^-)\, dx_1 = 2E(u,v;\Omega) \tag{1.8}$$

Here Ω is the subdomain $\mathbf{R}^2 \backslash \Lambda_-$, ds is an element of length of the arc, $\nu = (\nu_1, \nu_2)$ is the unit vector of the outward normal, and $\sigma^{(\nu)} = (\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \sigma_3^{(\nu)})$. Moreover

$$L(\nabla)u(x) = L^{\pm}(\nabla)u^{\pm}(x), \quad x \in \Omega_{\pm} \equiv \Omega \cap \mathbf{R}^2_{\pm} \tag{1.9}$$

$$\sigma_j^{(\nu)}(u;x) = \nu_1(x)\sigma_{j1}^{\pm}(u^{\pm};x) + \nu_2(x)\sigma_{j2}^{\pm}(u^{\pm};x)$$

$$x \in (\partial\Omega)_{\pm} \equiv \{x \in \partial\Omega : \pm x_2 \geqslant 0\}$$

Problem (1.5)–(1.7) is linear, that is, contact between the sides is ignored even in the case where there are oscillating singularities corresponding to complex λ in (1.1), and non-linear contact conditions are required in order to remove any overlapping of the sides (see [1–3], etc.). Nevertheless, thanks to the results in [4–6], which localize non-linear effects, the problem is well understood and there have been a large number of papers concerning the investigation of singularities of its solutions ([4–19], etc.). In the main, problem (1.5)–(1.7) has been split into a plane and an antiplane problem and direct analytic computations have been performed. In this paper we use the approach of [20] (see also [21, Ch. 7]), which is valid for general self-conjugate systems with piecewise-constant coefficients and helps to avoid routine computations.

2. DIFFERENTIATION ALONG THE CRACK

We will first give some known facts from the theory of elliptic problems in regions with corner (conical) points ([22–24] and [21, 25, Ch. 3]). The number λ in (1.1) is characteristic for the problem on the arc $(-\pi, \pi)$, obtained by substituting the vector $r^{\lambda}\Phi(\varphi)$ into (1.5)–(1.7). We will use the symbol $A(\lambda)$ for the operator of this problem for brevity. The abstract function $\lambda \mapsto A(\lambda)$ (bundle) is quadratic. The Jordan chain (JC) $\Psi^0, \ldots, \Psi^{\varkappa-1}$ of length $\varkappa$, corresponding to the characteristic value (CV) λ_* of bundle A consists of the eigenvector Ψ^0 and adjoint vectors $\Psi^1, \ldots, \Psi^{\varkappa-1}$, which are found from the problems

$$A(\lambda_*)\Psi^k = -\sum_{p=1}^{k} \frac{1}{p!} \frac{d^p A}{d\lambda^p}(\lambda_*)\Psi^{k-p}, \quad k = 0, \ldots, \varkappa - 1 \tag{2.1}$$

Each JC corresponds to the set of PS

$$V^k(x) = r^{\lambda_*} \sum_{p=1}^{k} \frac{1}{p!}(\ln r)^p \Psi^{k-p}(\varphi), \quad k = 0, \ldots, \varkappa - 1 \tag{2.2}$$

Any PS (1.1) can be represented in the form (2.2), where $\Psi^0, \ldots, \Psi^k$ is a JC corresponding to the CV λ_*.

We denote by Σ the spectrum of the bundle A or, in other words, the set of indices in non-trivial PS (1.1) of problem (1.5)–(1.7). In view of the fact that the coefficients of the operators $L^{\pm}$ and $N^{\pm}$ are real

$$\lambda \in \Sigma \Rightarrow \bar{\lambda} \in \Sigma \tag{2.3}$$

The bar denotes the complex conjugate. Problem (1.5)–(1.7) is formally self-conjugate, that is, according to [25], Section 5.5 and [21, Section 6.1], we have

$$\lambda \in \Sigma \Rightarrow -\bar{\lambda} \in \Sigma \tag{2.4}$$

The coefficients of the operators are constant and the vector $\partial_1 U$, obtained by differentiating (1.1) with respect to x_1, also satisfies (1.5)–(1.7). Since $\varphi \mapsto \Phi(\varphi, \ln r)$, which is a smooth function for $\varphi \neq 0$, equality to $\partial_1 U$ is possible if, and only if, $U^{\pm}(x) = x_2^k a^{\pm}$. It can be verified by direct computation that Eq. (1.5) is not satisfied when $k \geqslant 2$, and condition (1.6) does not hold when $k = 1$. There is still the obvious possibility: $U^{\pm} = a$—a rigid shift. Thus

$$0 \in \Sigma; \; \lambda \in \Sigma \backslash 0 \Rightarrow \lambda - 1 \in \Sigma \tag{2.5}$$

3. THE POLYNOMIAL PROPERTY OF THE ENERGY FUNCTIONAL

For form (1.3) we have the inequality

$$E^{\pm}(u^{\pm},u^{\pm};\Omega_{\pm})\geqslant c_{\pm}\left\|\varepsilon(u^{\pm});L_2(\Omega_{\pm})\right\|^2 \tag{3.1}$$

The strain tensor and, therefore, the left-hand side of (3.1), degenerates only on rigid shifts ($c_1 - c_0x_2$, $c_2 + c_0x_1$). This is the polynomial property of the quadratic form (1.4) ([25, Section 5.1] and [21, Sections 5.4 and 6.1]); it gives much useful information concerning problem (1.5)–(1.7). Thus, by Theorem 5.5.2 of [25] or 6.1.2 of [21] the CV $\lambda = 0$ corresponds to the three eigenvectors (EV)

$$\Phi^{1,0}=e^1=(1,0,0),\ \ \Phi^{2,0}=e^2=(0,1,0),\ \ \Phi^{3,0}=e^3=(0,0,1) \tag{3.2}$$

For each of these there is an adjoint $\Phi^{j,1}$. There is no JC of length greater than two, and no other CV on the imaginary axis $i\mathbf{R} = \{\lambda \in \mathbf{C}\colon \operatorname{Re}\lambda = 0\}$. The PS $V^{j,1}$ found for $\Phi^{j,0}$ and $\Phi^{j,1}$ with the help of (2.2), correspond to forces concentrated at the tip $x = 0$ of the crack Λ_-.

Operators with smooth coefficients were analysed in [25, 21], whereas the coefficients of the operator $L(\nabla)$ have a discontinuity on Λ_+. This does not affect the proof of the statements made, since the interface conditions (1.7) are worked out exactly the same way as boundary conditions (1.6).

From the above, it can be concluded that, for integer m, there is only one CV $\lambda = m$ on the straight line $m + i\mathbf{R} = \{\lambda \in \mathbf{C}\colon \operatorname{Re}\lambda = m\}$, that is

$$\lambda\in\Sigma,\ \ \operatorname{Re}\lambda=m\in\mathbf{Z}\Rightarrow\lambda=m \tag{3.3}$$

It will be proved in Section 5 that the CV $\lambda = m \in \mathbf{Z}\backslash 0$ correspond to exactly three linearly independent PS (for $m = 0$ there are six $V^{j,0} = \Phi^{j,0}$ and $V^{j,1} = \Phi^{j,0}\ln r + \Phi^{j,1}$; $j = 1, 2, 3$). It can be verified that any PS generated by the CV $\lambda = m < 0$ is a combination of derivatives $\partial_1^m V^{j,1}$, and the power series with index $\lambda = m > 0$ is a vector polynomial.

4. BIORTHOGONALITY CONDITIONS

We will now show that, apart from integer points, the spectrum lies only on straight lines $i\mathbf{R} + n + 1/2$, where $n \in \mathbf{Z}$. By virtue of (2.5) and (2.3), it is sufficient to establish that the strips $\Pi_0 = \{\lambda \in \mathbf{C}\colon 0 < \operatorname{Re}\lambda < 1/2\}$ and $\Pi_1 = \{\lambda \in \mathbf{C}\colon 1/2 < \operatorname{Re}\lambda < 1\}$ are free from the spectrum. To verify this, we need information of a general kind ([2, 3]), as well as [25, 21, Section 3.5]). We know that for any (non-trivial!) PS

$$U(x)=r^{\lambda}\sum_{p=0}^{s}\frac{1}{p!}(\ln r)^p\Phi^{s-p}(\varphi) \tag{4.1}$$

there is a power series $V^n(x)$ of the form (2.2) with $\lambda_* = -\bar{\lambda}$ such that

$$Q(U,V^n;\Gamma)\equiv\int_{\Gamma}\{\overline{V^n(x)}\sigma^{(\nu)}(U;x)-U(x)\overline{\sigma^{(\nu)}(V^n;x)}\}ds=1 \tag{4.2}$$

In addition to (4.2), for the other PS of (2.2) we have

$$Q(U,V^k;\Gamma)=0,\ \ k=0,\dots,n-1,n+1,\dots,\varkappa-1 \tag{4.3}$$

and in the case $s = 0, \varkappa > 1$ necessarily $n > 0$. In (4.2) and (4.3) Γ is any piecewise-smooth path joining the opposite sides of the cut Λ_- and not intersecting itself. Finally, if U and V satisfy (1.5)–(1.7) near Γ, then ([20] and [21, Lemma 7.4.4])

$$Q(\partial_1 U,V;\Gamma)=-Q(U,\partial_1 V;\Gamma) \tag{4.4}$$

Let the CV λ be in the strip Π_0 (or in Π_1). We take the corresponding PS (4.1). The derivative $\partial_1 U$ is also a non-trivial PS and is generated by the CV $\lambda - 1$. We seek one more PS with respect to $\partial_1 U$

$$V(x)=r^{1-\bar{\lambda}}Y(\varphi,\ln r) \tag{4.5}$$

subject to the condition (cf. (4.2) and (4.4))

$$1 = Q(\partial_1 U, V; \Gamma) = Q(\partial_1 V, U; \Gamma) \tag{4.6}$$

where the CV $1 - \bar{\lambda}$ lies in Π_1 (or in Π_0) and, therefore, $U \neq V$. Thus, we have constructed two pairs of power series $\partial_1 U$, V and $\partial_1 V$, U, related by the biorthogonality conditions (4.6).

In Section 8, when considering the lengthening of the crack by an amount h, we will see the following: if, in the initial position of the crack, the solution is identical with the sum

$$c_U U(x) + c_V V(x) \quad (c_U, c_V \in \mathbf{R}) \tag{4.7}$$

then the increment of potential energy $\Delta\mathrm{U}$ satisfies the relation

$$\Delta\mathrm{U} = -hc_U c_V + o(h) \tag{4.8}$$

It is important that only three conditions need to be satisfied in order to derive (4.8): relation (4.6), the inequality $U \neq V$ and the inclusion $\operatorname{Re}\lambda \in (0, 1)$.

According to (4.8), the increment $\Delta\mathrm{U}$ can be given any sign depending on the choice of c_U and c_V in (4.7). At the same time, $\Delta\mathrm{U} \leqslant 0$, since the solution of the problem gives a minimum of the energy functional, and as the crack grows, the functional space on which the minimum is sought expands. The resulting discrepancy contradicts the assumption made, and thus

$$\Sigma \cap \Pi_0 = \Sigma \cap \Pi_1 = \phi \tag{4.9}$$

A similar argument can be used to check that the CF is algebraically simple. Let $\operatorname{Re}\lambda = 1/2$, $U(x) = r^{\lambda\Phi}(\varphi)$, but $\varkappa > 1$, that is, let there be a PS with a logarithm. Then $\partial_1 U(x) = r^{\lambda-1}\Theta(\varphi)$ and, therefore, in the PS (4.5), biorthogonal to $\partial_1 U$, $1 - \lambda = \lambda$ and Υ is a polynomial of non-zero degree (see the comments concerning (4.3)). Thus, $U \neq V$, as required.

We will show that the total (total algebraic) multiplicity of the CV on each of the straight lines $i\mathbf{R} + l/2$, where $l \in \mathbf{Z}$, $l \neq 0$, is three (we recall that $\lambda = 0$ has multiplicity 6). Thus we will fill up the gap in the proof of Section 3 and, furthermore, by virtue of (2.3)–(2.5), we have

$$\Sigma = \{n, n + {}^1\!/_2, \pm i\gamma + n + {}^1\!/_2 \colon n \in \mathbf{Z}\} \tag{4.10}$$

where $\gamma \geqslant 0$. As usual, it is sufficient to confine ourselves to considering the straight lines $i\mathbf{R} + 1/2$ and $i\mathbf{R} + 1$.

If the half-planes $\mathbf{R}^2_\pm$ are filled with the same isotropic material, we know that $\lambda = 1/2$ and $\lambda = 1$ are triple CV, and the PS for $\lambda = 1/2$ generate radical singularities of stresses of three modes. When $\lambda = 1$ the PS correspond to a rotation about the axis $x_3 = 0$ and two single-axis loadings parallel to the crack (plane and antiplane problems).

By a continuous change of the elastic moduli, one can change from an isotropic to any homogeneous anisotropic plane with a cut and then, by varying the moduli only in the lower half-plane, to problem (1.5)–(1.7). Since the full algebraic multiplicity of the CV is presented during a completely continuous perturbation of the bundle (the exact formulation of the theorem is given in [26, Section 1.3]) and as proved above, the CV cannot leave their straight lines or go to infinity (the given transformation is parametrized by points $t \in [t_0, t_1]$), if the bundle $\mathcal{A}$ occurring (2.1) is given an exact meaning, the above assertion can be stated as a theorem.

We recall the generalized formulation of the problem in region Ω (see Fig. 1)

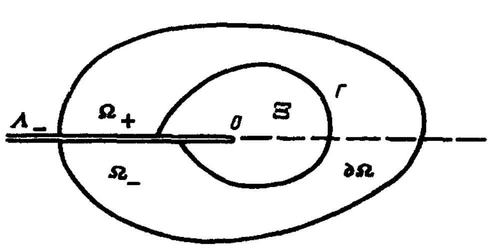

Fig. 1.

$$L(\nabla)\, u^{\pm}(x) = F(x),\ x \in \Omega_+ \cup \Omega_- \tag{4.11}$$

$$\sigma^{(\nu)}(u;x) = G(x),\ x \in (\partial\Omega)_+ \cup (\partial\Omega)_- \tag{4.12}$$

$$u^+(x) = u^-(x),\ N^+(\nabla)u^+(x) - N^-(\nabla)u^-(x) = H(x),\ x \in \Lambda_- \cap \Omega \tag{4.13}$$

By a generalized solution, we mean an element u of the Sobolev space $W_2^1(\Omega)^3$ which satisfies the integral identity

$$2E(u,v;\Omega) = \int_\Omega (f^0 v + \sum_{k=1}^{2} f^k \partial_k v)\,dx + \int_{\partial\Omega} gv\,ds + \int_{\Lambda_-\cap\Omega} hv\,dx_1,\ \ \forall v \in W_2^1(\Omega)^3 \tag{4.14}$$

$$F = f^0 - \sum_{k=1}^{2} \partial_k f^k,\ \ G = g + \sum_{k=1}^{2} n_k f^k,\ \ H = h + f^{2-} - f^{2+}$$

$$f^{q\pm} \in L_2(\Omega_\pm),\ \ g \in L_2(\partial\Omega),\ \ h \in L_2(\Lambda_+ \cap \Omega)$$

The generalized formulation of the problem is transferred in the usual way to the bundle $\lambda \mapsto A(\lambda)$ with domain of definition $W_2^1(-\pi,\pi)^3$ (for details, see [21–25], for example). The next bundle possesses the same CV and JC

$$1 + B(\lambda) = 1 + A(\lambda_0)^{-1}(A(\lambda) - A(\lambda_0)) : W_2^1(-\pi,\pi)^3 \to W_2^1(-\pi,\pi)^3 \tag{4.15}$$

In (4.15) 1 is the identity operator, $\lambda_0 \in \mathbb{C}\backslash\Sigma$ (for example, $\lambda_0 = 1/4 \in \Pi_0$), and $B(\lambda)$ is an entirely continuous operator on $W_2^1(-\pi,\pi)^3$ (since the coefficients of the highest derivatives in the system of equations on $(-\pi,\pi)$ in the boundary conditions and in the conjugation conditions are independent of λ and, therefore, disappear in the difference $A(\lambda) - A(\lambda_0)$). In fact, B undergoes changes on changing from an isotropic to a composite plate.

5. THE ESHELBY-CHEREPANOV-RICE INTEGRAL AND SPLITTING OF THE STRESS-STRAIN STATE INTO MODES

In the crack theory wide use is made of the invariant integral (independent of the path) [27–29]

$$J(u;\Gamma) = \int_\Gamma \{W(u,\bar{u};x)\cos(\nu,x_1) - \partial_1 u(x)\cdot\sigma^{(\nu)}(u;x)\}\,ds \tag{5.1}$$

Here $\cos(\nu, x_1)$ is the direction cosines of the outward normal ν to the arc Γ, and $W^{\pm}$ are the densities of elastic energies $E^{\pm}$. Provided that F, G and H are equal to zero near Γ, the solution u of problem (4.11)–(4.13) satisfies the equation ([20 and 21, Section 7.3])

$$J(u;\Gamma) = 2^{-1}Q(\partial_1 u, u; \Gamma) \tag{5.2}$$

Formula (5.2) can be used to evaluate the integral (5.1). Let Ξ denote a subdomain of Ω which contains the arc Γ and the set which it bounds (it is completed by segments of the crack sides). Suppose that

$$F(x) = 0,\ x \in \Xi;\ G(x) = 0,\ x \in \partial\Omega \cap \partial\Xi;\ H(x) = 0,\ x \in \Lambda_+ \cap \Xi \tag{5.3}$$

The solution $u \in W_2^1(\Omega)^3$ of problem (4.11)–(4.13) is given by the formula

$$u(x) = c + \sum_{\omega=0,\pm} K_\omega r^{i\gamma\omega + 1/2}\Phi^\omega(\varphi) + O(r) \tag{5.4}$$

Here K_0, $K_\pm$ are the stress intensity factors (SIF), Φ^0, $\Phi^\pm$ are the EV corresponding to the CV 1/2, $\pm i\gamma + 1/2$ and

$$U^\tau(x) = r^{i\gamma\tau}\Phi^\tau(\varphi)\ \ (\tau = 0, \pm) \tag{5.5}$$

Since $J(u;\Gamma) = J(u - c;\Gamma)$, it can be assumed in (5.4) that $c = 0$. Contracting the contour Γ to the tip, we get rid of the residue $O(r)$. Thus, using (4.4) and (4.2) we have

$$J(u;\Gamma) = \frac{1}{2}\sum_{\tau,\varkappa=0,\pm} \bar{K}_\tau M_{\tau\varkappa} K_\varkappa \tag{5.6}$$

$$M_{\tau\varkappa} = Q(\partial_1 U^\varkappa, U^\tau; \Gamma) = -\overline{Q(U^\tau, \partial_1 U^\varkappa; \Gamma)} = \overline{M_{\varkappa\tau}} \tag{5.7}$$

The matrix M is symmetric and positive definite (cf. (5.7) and [20] and [21, Section 7.4]).

Apart from (4.2) and (4.3), there is another property of the PS that should be pointed out [25 and 21, Sections 3.2.1 and 3.5.4]: if U and V correspond to the EV λ and λ_2, but $\lambda_2 - \lambda_1$, then

$$Q(U, V; \Gamma) = 0 \tag{5.8}$$

Thus, M is a diagonal real matrix when $\gamma \neq 0$. Also, $\Phi^\pm$ can be taken as complex conjugates, and we have $M_{++} = M_{--}$. We will put

$$U^1 = 2^{-1/2}(U^+ + U^-), \quad U^2 = i2^{-1/2}(U^+ - U^-), \quad U^3 = U^0$$
$$K_1 = 2^{-1/2}(K_+ + K_-), \quad K_2 = -i2^{-1/2}(K_+ - K_-), \quad K_3 = K^0 \tag{5.9}$$

Now U^j are real and $K_j \in \mathbf{R}$ for real F, G and H (we have removed the complex SIF; cf. [11, 13]). Moreover, by virtue of (5.8), (5.7) and (5.9)

$$m_p = Q(\partial_1 U^p, U^p; \Gamma) = 2^{-1}[Q(\partial_1 U^+, U^+; \Gamma) + Q(\partial_1 U^-, U^-; \Gamma)] =$$
$$= 2^{-1}(M_{++} + M_{--}) = M_{++} > 0 \ (p = 1, 2) \tag{5.10}$$
$$Q(\partial_1 U^1, U^2; \Gamma) = Q(\partial_1 U^2, U^1; \Gamma) = 0, \quad m_3 = M_{00} > 0$$

Thus, formulae (5.4) and (5.6) in the notation (5.9) take the form

$$u(x) = c + \sum_{j=1}^{3} K_j U^j(x) + O(r) \tag{5.11}$$

$$J(u; \Gamma) = \frac{1}{2} \sum_{j=1}^{3} m_j K_j^2 \tag{5.12}$$

Each basis in the lineal $\mathbf{L}_{1/2}$ of the PS with indices Re $\lambda = 1/2$ is accompanied by its own splitting of a radical singularity of the stressed state into modes and, therefore, by its own SIF vectors and matrix M. When $\gamma \neq 0$ the basis, complex $\{U^+, U^-, U^0\}$ or real $\{U^1, U^2, U^3\}$, is defined in (5.5) and (5.9) is fully single-valued. When $\gamma = 0$ all the PS of $\mathbf{L}_{1/2}$ are generated by the same CV $\lambda = 1/2$ and it is therefore impossible to define a canonical basis when operating only with the bundle A.

We will now show how to adjust the choice of bases to force or energy criteria of fracture.

Let $\gamma = 0$ and $\{U^j\}$ be any basis in $\mathbf{L}_{1/2}$. We form a numerical 3×3 matrix Σ with elements

$$\Sigma_{kj} = 2^{-1} r^{1/2} \{\sigma_{2k}^+(U^j; r, +0) + \sigma_{2k}^-(U^j; r, -0)\} \tag{5.13}$$

The matrix Σ is non-singular. For let $\Sigma\, b = 0$ for some column b. The PS $U = b_j U^j$ (with index $\lambda = 1/2$) satisfies (1.5) on $\{x: x_2 = 0\}$ and (1.6) on $R^2_\pm$ (since according to (1.2) and (5.13), (1.7) $N^\pm(\nabla)$ $U(x) R^{-1/2} \Sigma\, b = 0$ for $x \in \Lambda_+$). At the same time, all the PS of the problem in the half-plane have integer indices ([30, Section 2] and [21, Section 6.4]) and therefore $U = 0$ and $b = 0$, that is, det $\Sigma \neq 0$.

We will denote the elements of the matrix inverse to Σ, Σ^{-1}, by $(\Sigma^{-1})_{qp}$ and introduce a new basis $\{U^{0p}\}$ in $\mathbf{L}_{1/2}$ by the equations

$$U^{0p} = (2\pi)^{-1/2}\{(\Sigma^{-1})_{1p} U^1 + (\Sigma^{-1})_{2p} U^2 + (\Sigma^{-1})_{3p} U^3\}$$

The only difference between the matrix $\Sigma^0 \equiv \Sigma$, found with respect to this basis, and the identity matrix is the presence of the factor $(2\pi)^{-1}$. Thus, by rewriting series (5.11) we obtain the classical definition of the SIF

$$u(x) = c + \sum_{p=1}^{3} K_{0p} U^{0p}(x) + O(r) \tag{5.14}$$

$$\sigma_{2p}(u;r,0)=2^{-1}\{\sigma_{2p}^{+}(u;r,+0)+\sigma_{2p}^{-}(u;r,-0)\}=(2\pi r)^{-1/2}K_{0p}+O(1),\quad r\to+0 \tag{5.15}$$

Since the stresses $r^{-1/2}\Sigma_{kj}$ appeared in the normalization conditions on the continuation of the crack, the force criterion must be used for choosing the basis $\{U^{0p}\}$.

Relation (5.6), transformed using (5.14), takes the form

$$J(u;\Gamma)=\frac{1}{2}\sum_{q,p=1}^{3}K_qM_{qp}^{0}K_p \tag{5.16}$$

Let S be a unitary 3×3 matrix which reduces the matrix $M^0=(M_{qp}^0)$ to diagonal form (its CV are $m_j>0$)

$$SM^0S^{-1}=\mathrm{diag}\{m_1,m_2,m_3\} \tag{5.17}$$

We will construct a new basis with respect to S and $\{U^{0p}\}$

$$U^j=S_{j1}U^{01}+S_{j2}U^{02}+S_{j3}U^{03}\quad(j=1,2,3) \tag{5.18}$$

Formulae (5.14) and (5.16) take the form (5.11) and (5.12). The relation between the new SIF K_j and K_{0p} is again given by (5.18) and in view of the fact that S is unitary, the moduli of the SIF vectors $K=(K_1,K_2,K_3)^t$ and $K_0=(K_{10},K_{20},K_{30})^t$ are equal (t denotes the transpose). Furthermore, for quantities (5.13) calculated with respect to the basis (5.18), by virtue of (5.15) we have

$$|\Sigma_{1j}|^2+|\Sigma_{2j}|^2+|\Sigma_{3j}|^2=2\pi \tag{5.19}$$

In Section 7, expression (5.12) will be interpreted as the increment of potential energy, that is, the basis (5.18) is associated with the energy criterion of fracture. If the CV m_1, m_2 and m_3 differ, S and $\{U^j\}$ are single-valued. But if, for example, $m_1=m_2\neq m_3$, S is defined to within a unitary multiplier, leaving the EV of the matrix M^0 corresponding to its CV m_3 unchanged. This arbitrariness in the choice of the "energy" basis of other considerations. Thus, for a homogeneous isotropic plate $m_1=m_2=\mu^{-1}(1-\nu)$ and $m_3=\mu^{-1}$, where μ and ν are the shear modulus and Poisson's ratio, and the "force" and "energy" bases are the same.

If $\gamma\neq0$ in (4.10), then by (5.10) $m_1=m_2$, and thus we have the two bases (5.5) and (5.9); these are related by a unitary transformation and, bearing in mind formula (5.12), are equivalent.

We will show what happens to the representation (5.11) when there is a change of scale. Let $h>0$ be a dimensionless parameter and

$$(\xi_1,\xi_2)=\xi=h^{-1}x=(h^{-1}x_1,h^{-1}x_2) \tag{5.20}$$

The replacement $x\mapsto\xi$ is accompanied by transformation of the 3×3 matrix $U(x)$, containing columns $U^j(x)$ of (5.10)

$$\sum_{j=1}^{3}K_jU^j(x)=U(x)K=U(h\xi)K=h^{1/2}U(\xi)\Theta^{-1}K \tag{5.21}$$

$$\Theta=\begin{vmatrix}c & s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{vmatrix},\quad \begin{matrix}c=\cos[\gamma\ln h]\\ s=\sin[\gamma\ln h]\end{matrix} \tag{5.22}$$

If the basis $\{h^{-1}U^j(\xi)\}$ is used, the new SIF vector takes the form $h^{1/2}\Theta^{-1}K$. However, in the usual situation $\gamma=0$ the rescaling of the SIF reduces to multiplication by $h^{-1/2}$; to preserve this rule, we need to take the columns of the matrix $h^{-1}U^j(\xi)\,\Theta^{-1}$ as the basis. This transformation is admissible in the energy approach (since $m_1=m_2$, the matrices $m=\mathrm{diag}\,\{m_1,m_2,m_3\}$ and Θ^{-1} commute, and there is no change in the form of formula (5.12)). The classical definition of the SIF (the force approach) in the case $\gamma\neq0$ is discussed in [6], where the need to introduce an additional structural parameter r_s which fixes the scale is indicated. A similar parameter is needed in definition (5.9): U^j must be so chosen that the quantities

$$\Sigma_{jk}^{s}=r_s^{1/2}\{\sigma_{2k}^{+}(U^j;r_s,+0)+\sigma_{2k}^{-}(U^j;r_s,-0)\} \tag{5.23}$$

satisfy (5.19). When $\gamma = 0$ the right-hand side of (5.23) is independent of r_s, and formulae (5.23) and (5.13) are indistinguishable.

6. THE GRIFFITHS FORMULA IN THE CASE WHEN THE CRACK BRANCHES

We will consider problem (4.11)–(4.13) under assumptions (5.3), corresponding to the deformation potential energy functional (we use the Betti identity (1.8))

$$\mathbf{U} = \mathbf{E} - \mathbf{A} = -2^{-1}\mathbf{A} \tag{6.1}$$

$$\mathbf{A} = \int_{\Omega} F u dx + \int_{\partial\Omega} G u ds + \int_{\Lambda_- \cap \Omega} H u dx_1 \tag{6.2}$$

Here $\mathbf{E}$ is the elastic energy and $\mathbf{A}$ is the work of external forces.

We will suppose that as a result of fracture the crack has acquired N branches, making angles θ_n with the x_1 axis. The case $N = 1$ is not excluded. Let the lengths of the branches be $h_n = hl_n$ and let the quantities $l_1, \dots, l_n$ be comparable with the characteristic length of the domain Ω; $h > 0$ is a small parameter. We will study the asymptotic behaviour of the solution of problem (4.11)–(4.13) in the singularly perturbed domain Ω_h, which contains a crack with branches. We will assume that their sides are stress-free, and the loading $\{F, G, H\}$ is the same as before. We will denote the functionals of (6.1), found by solving the problem in Ω_h, by $\mathbf{U}_h$, $\mathbf{E}_h$, $\mathbf{A}_h$. We introduce the "fast" variables ξ by Eqs (5.21). After the change to $h = 0$, the region Ω_h, written using coordinates (5.21), transforms to the composite plane $\omega = \omega_+ \cup \omega_-$, weakened by a semi-infinite cut Λ_- and the cracks $\Lambda_n = \{\xi: \varphi - 0_n, \rho \in [0, l_n]\}$ coming from its tip; here (ρ, φ) are polar coordinates, $\rho = |\xi| = h^{-1}r$. With the branch Λ_n, we will associate Cartesian coordinates ξ^n and polar coordinates (ρ_n, φ_n) with centre at its end $P^n = (l_n \cos\theta_n, l \sin\theta_0)$; $\varphi_n \in (-\pi, \pi)$; $\xi_1^n = \rho_n \cos\varphi_n$, $\xi_2^n = \rho_n \sin\varphi_n$. Finally, $\tau^n = (\cos\theta_n, \sin\theta_n)$ and $\tau^n \cdot \nabla_\xi = \cos\theta_n \partial_2 = \partial/\partial\xi_1^n$.

Intending to apply the method of matched asymptotic expansions (cf. [21, 31–33], etc.), we will describe a number of special solutions of the limiting problems in Ω and ω. Weight functions which are singular solutions of the homogeneous problem (4.11)–(4.13)

$$\zeta^j = m_j^{-1}\partial_1 U^j + \zeta^{j0} \tag{6.3}$$

have already been introduced ([19, 23, 34], see also [21, 25, Section 4.3]). Here U^j is taken from (5.12), and $\zeta^0 \in W_2^1(\Omega)^3$ is the regular part of ζ^j.

Using the Betti identity and formulae (5.7), (5.10) and (5.17), we obtain the well-known integral representation of the SIF

$$\int_{\Omega} F\zeta^j dx + \int_{\partial\Omega} G\zeta^j ds + \int_{\Lambda_+ \cap \Omega} H\zeta^j dx_1 =$$

$$= -Q(u, \zeta^j; \Gamma) = -Q\left(\sum_{i=1}^{3} K_i U^i, m_j^{-1}\partial_1 U^j; \Gamma\right) = K_j \tag{6.4}$$

$$-Q(U^j, \partial_1 U^k; \Gamma) = Q(\partial_1 U^k, U^j; \Gamma) = m_j \delta_{j,k} \tag{6.5}$$

The analogous singular solutions of the problem in ω are the vectors

$$\eta^j(\xi) = U^j(\xi) + \eta^{j0}(\xi) = U^j(\xi) + \sum_{k=1}^{3} N_{jk} m_k^{-1}\partial_1 U^k(\xi) + Q(\rho^{-1}), \quad \rho \to \infty \tag{6.6}$$

In (6.6) $\partial_1 = \partial/\partial\xi_1$, and the real 3×3 matrix $N = (N_{jk})$ is symmetric and positive definite, since it is a Gram matrix

$$E(\eta^{j0}, \eta^{k0}; \omega) = \int_{\partial\omega} \sigma^{(\nu)}(\eta^{j0})\eta^{k0} ds = \int_{\partial\omega} \sigma^{(\nu)}(\eta^{j0})\eta^k ds =$$

$$= Q(\eta^{j0}, \eta^{k0}; \Gamma) = Q\left(\sum_{p=1}^{3} N_{jp} m_p^{-1}\partial_1 U^p, U^k; \Gamma\right) = N_{jk} \tag{6.7}$$

We now perform the matching procedure. It is convenient to use the 3×3 matrices U, ζ, η with columns U^i, ζ^i, η^i and also m and Θ. We will take the basic term of the outer series (far from the tip O) of the solution u^h of the singularly perturbed problem as the solution $u \in W_2^1(\Omega)^3$ of problem (4.10)–(4.13). Using (5.21), we rewrite (5.11) in the form

$$u(x) = c + U(x)K + O(r) = c + h^{1/2}U(\xi)\Theta^{-1}K + O(h\rho) \tag{6.8}$$

The principal terms of the inner series, which cater for the same vector function u^h near the point O and use scale ξ, as $\rho = |\xi| \to \infty$ necessarily behave in agreement with the asymptotic terms on the right of (6.8). Thus it is reasonable to start this series with the sum

$$c + h^{1/2}\eta(\xi)\Theta^{-1}K \tag{6.9}$$

By virtue of (6.6) it can be represented as

$$c + h^{1/2}\left[U(\xi) + \frac{\partial U}{\partial \xi_1}(\xi)m^{-1}N\right]\Theta^{-1}K + O(h^{1/2}\rho^{-1}) \tag{6.10}$$

Applying (5.21), with h, ξ replaced by h^{-1}, x, we return to the slow variables x and instead of (6.10) obtain the new representation

$$c + U(x)K + h\frac{\partial U}{\partial x_1}(x)\Theta m^{-1}N\Theta^{-1}K + O(h^{3/2}\rho^{-1}) \tag{6.11}$$

The first two terms of (6.11) appear in the middle of (6.8), and the third fixes the behaviour, near the tip, of the coefficient of h in the outer series. As already mentioned, the matrices Θ and m^{-1} commute, that is, the third term in (6.11) is equal to

$$h\partial_1 U(x)\, m^{-1}\Theta N\Theta^{-1}K \tag{6.12}$$

Now recalling the weight functions (6.4), which possess the same singularities as (6.14), we improve the accuracy of the outer series with the second term

$$u(x) + h\,\zeta(x)\,\Theta N\,\Theta^{-1}K \tag{6.13}$$

Higher-order corrections are unnecessary in either series (6.9) or (6.13).

The residues in the asymptotic form are found in the usual way ([33, Ch. 5] and [21, Section 6.5]), but after that, only the error of the approximate formula for the energy increment is indicated below

$$\Delta\mathbf{U} = \mathbf{U}_h - \mathbf{U} = -2^{-1}(\mathbf{A}_h - \mathbf{A}) =$$
$$= -\frac{1}{2}\int_{\Omega} F(u^h - u)\,dx - \frac{1}{2}\int_{\partial\Omega} G(u^h - u)\,ds - \frac{1}{2}\int_{\Lambda_+\cap\Omega} H(u^h - u)\,dx_1 \tag{6.14}$$

Since the sum (6.13) approaches u^h far from O, using it to replace u^h in (6.14) and taking (6.4) into account, we arrive at the require result

$$\Delta\mathbf{U} = -2^{-1}hK^t\Theta N\Theta^{-1}K + O(h^{3/2}) \tag{6.15}$$

7. THE GRIFFITHS FORMULAE REWRITTEN

Even if $\gamma = 0$, when Θ is the identity matrix, relation (6.15) is unsatisfactory owing to the presence of the matrix N. It would be better if the form of (6.15) involved quantities for which there is a physical interpretation.

Each of the branch-cracks Λ_n is associated with three PS U^{jn} corresponding to CV on $i\mathbf{R} + 1/2$, as well as the numbers γ_n and $m_1^{(n)}, m_2^{(n)}, m_3^{(n)}$. Furthermore, we have the representations

$$\eta^p(\xi) = c^p + \sum_{j=1}^{3}\mathbf{K}_j^{pn}U^{jn}(\xi^n) + O(\rho_n), \quad \rho_n \to 0 \tag{7.1}$$

We will construct columns $(\mathbf{K}_1^{pn}, \mathbf{K}_2^{pn}, \mathbf{K}_3^{pn})^t$, and use them to form the matrix $\mathbf{K}^{(n)}$. As in Section 6, we introduce matrices $U^{(n)}$, $m^{(n)}$ and $\Theta^{(n)}$. Then

$$c + h^{1/2}\eta(\xi)\Theta^{-1}K = c + h^{1/2}U^{(n)}(\xi^n)\mathbf{K}^{(n)}\Theta^{-1}K + O(h^{1/2}\rho_n) \tag{7.2}$$

Since approximation (6.9) holds for the neighbourhood of the point O which the cracks Λ_n reach, from (7.2) and (5.21) we see that the column $K^{(n)}(h)$ containing the SIF of the field u^h at the tip P^n, is given by the relation

$$K^{(n)}(h) = \Theta^{(n)}\mathbf{K}^{(n)}\Theta^{-1}K + O(h^{1/2}) \tag{7.3}$$

The residual term in (7.3) can be estimated as described in the proof of Theorem 7.2.7 in [21].

We will now find a relation between the matrices N and $\mathbf{K}^{(1)}, \ldots, \mathbf{K}^{(N)}$, using the invariant integral [29, 35, 36]

$$J_r(u;\Gamma) = \int_\Gamma \{W(u,u;x)x\,\nu(x) - \sigma^{(\nu)}(u;x)\,r\partial_r u(x)\}\,ds = 2^{-1}Q(r\partial_r u, u;\Gamma) \tag{7.4}$$

Replacing x and r in (7.4) by Ξ and ρ, substituting the vector (6.11) for u, we choose the arc Γ to start on the upper side of the cut Λ, to include all the branches $\Lambda_1, \ldots, \Lambda_N$ and ending on its lower side. We have

$$\rho\frac{\partial}{\partial\rho} = \xi\nabla_\xi = (l_n\tau^n + \xi^n)\nabla_\xi = l_n\frac{\partial}{\partial\xi_1^n} + \rho_n\frac{\partial}{\partial\rho_n} \tag{7.5}$$

The differential expression $\rho_n\partial/\partial\rho_n$ does not strengthen the singularity in (7.1) and so, replacing integration over Γ by integration over the circles S_n with centres P^n and radii $\delta > 0$, we let δ tend to zero and, using (7.5), (5.11) and (8.3), obtain

$$\begin{aligned} J_\rho(\eta\Theta^{-1}K;\Gamma) &= \frac{1}{2}\sum_{n=1}^{N} l_n Q\left(\frac{\partial}{\partial\xi_1^n}\eta\Theta^{-1}K, \eta\Theta^{-1}K; S_n\right) = \\ &= \frac{1}{2}\sum_{n=1}^{N} l_n K^t\Theta\mathbf{K}^{(n)t}m^{(n)}\mathbf{K}^{(n)}\Theta^{-1}K = \frac{1}{2}\sum_{n=1}^{N} l_n K^t\Theta\mathbf{K}^{(n)t}\Theta^{(n)t}m^{(n)}\Theta^{(n)}\mathbf{K}^{(n)}\Theta^{-1}K = \\ &= \frac{1}{2}\sum_{n=1}^{N} l_n K^{(n)}(h)^t m^{(n)} K^{(n)}(h) + O(h^{1/2}) \end{aligned} \tag{7.6}$$

We will now find a different expression for the invariant integral. By direct verification we see that

$$Q(\rho\partial_\rho U^j, \partial_1 U^k;\Gamma) + Q(\rho\partial_\rho\partial_1 U^j, U^k;\Gamma) = -m_j\delta_{j,k} \tag{7.7}$$

Now, by virtue of (5.8), (4.4) and (7.7)

$$\begin{aligned} 2J_\rho(\eta\Theta^{-1}K;\Gamma) &= -Q(\rho\partial_\rho U\Theta^{-1}K, \partial_1 U m^{-1}N\Theta^{-1}K;\Gamma) - \\ &- Q(\rho\partial_\rho\partial_1 U m^{-1}N\Theta^{-1}K, U\Theta^{-1}K;\Gamma) = K^t\Theta m m^{-1}N\Theta^{-1}K = K^t\Theta N\Theta^{-1}K \end{aligned} \tag{7.8}$$

Comparing (7.6) and (7.8), we reduce formula (6.15) to the form

$$\begin{aligned} \Delta\mathbf{U} &= -\frac{1}{2}h\sum_{n=1}^{N} l_n K^{(n)}(h)^t m^{(n)} K^{(n)}(h) + O(h^{3/2}) = \\ &= -\frac{1}{2}h\sum_{n=1}^{N} l_n \sum_{j=1}^{3} m_j^{(n)} K_j^{(n)}(h)^2 + O(h^{3/2}) \end{aligned} \tag{7.9}$$

It should be noted that (7.9) includes the lengths of the branch-cracks, the SIF at their tips and the multipliers $m_j^{(n)}$, characterizing the material and direction of crack propagation. The order $(O(h^2))$ of the remainder in the classical formula of Griffiths remains unchanged when K_j is replaced by $K_j(h)$, after which it is no different in form from (7.9). Finally, by repeating the previous calculations with very slight changes it can be shown that

$$\Delta \mathrm{U} = J_r(u^h;\Gamma) + O(h^{3/2})$$

8. APPENDIX

We will now fill up the gaps in the proofs of Sections 5 and 7 (formula (4.8) and inequalities $m_j > 0$). Using the argument of Section 6 for the situation of Section 4 we have the PS U and V corresponding to CV λ and $1-\lambda$, which are in $\Pi = \{\lambda \in \mathbf{C}: 0 < \operatorname{Re}\lambda < 1\}$, with $U \neq V$. Conditions (4.6) hold. Consider the field u defined by formula (5.7) and the weight functions

$$\zeta^U = \partial_1 U + \zeta^{U0}, \quad \zeta^V = \partial_1 V + \zeta^{V0} \tag{8.1}$$

A singular perturbation of region Ω consists of the crack growth by an amount h, so that the region ω is made up of half-planes $\mathbf{R}^2_\pm$ meeting along the ray $\Lambda_0 = \{\xi: \xi_1 = 0, \xi_2 > 1\}$. We introduce the Cartesian coordinates

$$\xi^0 = (\xi_1^0, \xi_2^0) = (\xi_1 + 1, \xi_2) \tag{8.2}$$

If $1-\bar{\lambda} = \lambda_*$ and $V = V^\tau$, $U = U^\varkappa$ (cf. (2.2)), then by changing from slow x to fast variables ξ we obtain

$$V^\tau(x) = r^{\lambda_*}\sum_{p=0}^{\tau}\frac{1}{p!}(\ln r)^p\Psi^{\tau-p}(\varphi) = h^{\lambda_*}\rho^{\lambda_*}\sum_{p=0}^{\tau}\frac{1}{p!}[\ln\rho + \ln h]^p\Psi^{\tau-p}(\varphi) =$$

$$= h^{\lambda_*}\rho^{\lambda_*}\sum_{k=0}^{\tau}\frac{1}{k!}(\ln h)^k\sum_{q=0}^{\tau-k}\frac{1}{q!}(\ln\rho)^q\Psi^{\tau-k-q}(\varphi) = h^{\lambda_*}\sum_{k=0}^{\tau}\frac{1}{k!}(\ln h)^k V^{\tau-k}(\xi) \tag{8.3}$$

$$U^\varkappa(x) = r^\lambda\sum_{p=0}^{\varkappa}\frac{1}{p!}(\ln r)^p\Phi^{\varkappa-p}(\varphi) = h^\lambda\sum_{j=0}^{\varkappa}\frac{1}{j!}(\ln h)^j U^{\varkappa-j}(\xi) \tag{8.4}$$

Applying the matching procedure we find, from (4.7) and (8.3), (8.4), that the principal terms of the inner series must be taken to be

$$W^h(\xi^0) = c_V h^{\lambda_*}\sum_{k=0}^{\tau}\frac{1}{k!}(\ln h)^k V^{\tau-k}(\xi^0) + c_U h^\lambda\sum_{j=0}^{\varkappa}\frac{1}{j!}(\ln h)^j U^{\varkappa-j}(\xi^0) \tag{8.5}$$

According to [37], from (8.2) we have

$$W^h(\xi^0) = W^h(\xi) + \partial_1 W^h(\xi) + \ldots \tag{8.6}$$

The dots denote higher derivatives (with respect to ξ_1) of the field $W^h(\xi)$ or a residue $o(\rho^{-1})$. We return in (8.6) to slow variables x and, reading the transformations (8.3) in reverse order with further arrangement of ∂_1, we obtain the relation

$$W^h(\xi^0) = c_U U(x) + c_V V(x) + h[c_U\partial_1 U(x) + c_V\partial_1 V(x)] + \ldots$$

Using (8.1), we take out two terms of the outer expansion

$$c_U U(x) + c_V V(x) + h[c_U\partial_1 U(x) + c_V\partial_1 V(x)]$$

Then (4.8) is obtained simply by repeating calculations (6.4) and (6.14)

$$\Delta \mathrm{U} = -2^{-1}hQ(c_U\partial_1 U + c_V\partial_1 V, c_U U + c_V V;\Gamma) + o(h) = -hc_U c_V + o(h)$$

Let $c_U = 1$, $c_V = 0$ and $U = K_+U^+ + K_-U^- + K_0U^0$. If the crack develops in a straight line, the special solutions (6.6) can be found explicitly: $\eta^\tau(\xi) = U^\tau(\xi^0)$, $\tau = 0, \pm$. By (8.6) $\eta^\tau(\xi) = U^\tau + \partial_1 U^\tau(\xi) + \ldots$. That the matrix M (and therefore the matrix m) is positive definite follows from the formula

$$\Delta \mathrm{U} = \frac{h}{2}Q(U,\zeta^U;\Gamma) = \frac{h}{2}Q\left(\sum_{\varkappa=0,\pm} K_\varkappa U^\varkappa, \sum_{\sigma=0,\pm} K_\sigma\partial_1 U^\sigma;\Gamma\right) +$$

$$+ o(h) = -\frac{h}{2}\sum_{\sigma,\varkappa=0,\pm}\bar{K}_\sigma M_{\sigma\varkappa}K_\varkappa + o(h)$$

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